ED-309

M.A./M.Sc. 1st Semester
Examination, March-April 2021

MATHEMATICS

Paper - I

Advanced Abstract Algebra - I

Time : Three Hours] [*Maximum Marks* : 80

Note : Answer any **two** parts from each question. All questions carry equal marks.

1. (*a*) Show that the symmetric group S_3 of degree 3 is solvable.

(*b*) Prove that a group of order P^n (P is prime) is nilopotent.

(*c*) Show that there exist atleast one composition series for each finite group.

2. (*a*) Let K be an extension of F. If a, b in E are algebraic over F, then prove $a \pm b$, ab and ab^{-1} ($b \neq 0$) are all algebraic over F.

(*b*) Let E be an extension field of F. If $a \in E$ is algebraic over F of odd degree, show that $F(a) = F(a^2)$.

(*c*) For any field K prove that the following statements are equivalent :

(*i*) K is algebraically closed.

(*ii*) Every irreducible polynomial in $K(x)$ is of degree 1.

(*iii*) Every polynomial in $K(x)$ of positive degree factors compeletely in $K(x)$ into linear factors.

3. (*a*) Let C be the field of complex numbers and R the field of real numbers. Show that C is a normal extension of R.

(*b*) If $f(x) \in F(x)$ is an irreducible polynomial over a finite field F, then show that all the roots of $f(x)$ are distinct.

(*c*) Let E be a finite extension of F, then E is a normal extension of F if and only if E is a splitting field of some polynomial over F.

4. (*a*) Let E be a finite normal extension of a field F. If α_1, α_2 are conjugate elements in E over F, then prove there exists an F-automorphism σ of E such that $\sigma(\alpha_1) = \sigma(\alpha_2)$.

(*b*) Let H be a finite subgroup of the group of automorphism of a field E. Then prove

$$|E : E_H| = |H|$$

(*c*) Find the Galois group of $x^3 - 2 \in Q(x)$.

5. (*a*) Prove that $f(x) \in F(x)$ is solvable by radicals over F if and only if its splitting field E over F has solvable Galois group $G(E/F)$.

(*b*) Show that the polynomial $2x^5 - 5x^4 + 5$ is not solvable by radicals.

(*c*) Show that if an irreducible polynomial $p(x) \in F(x)$ over a field F has a scot in a radical extension of F, then $p(x)$ is solvable by radicals over F.